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Distributed Event-Based Control for Second-Order Multi-Agent Systems

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Abstract— **In this paper, we propose a state dependent event triggering condition to achieve consensus for a team of agents modeled with second order integrator dynamics. We study two event-based consensus algorithms. In the first algorithm we guarantee that the agents reach position consensus with a constant final velocity while the second algorithm guarantees that the agents reach position consensus with a zero final velocity. The results are verified through simulation examples.**

I. INTRODUCTION

Multi-agent systems (MAS) are systems composed of multiple interacting dynamic units that are characterized by their autonomy, local perspective, and decentralization. These systems are recently gaining attention due to their broad applications in wireless sensor networks, formation flying, and distributed robotic systems [1]–[3]. MAS can either be centrally controlled by a central computer or distributedly controlled where every agent decides its own course of action. Coordination between systems is brought by solving the consensus problem, where consensus is defined as an agreement by multiple systems on a common state [4]. With the computation and modeling tasks becoming more and more complex, centralized systems are difficult to maintain and operate, hence a distributed approach is needed where the agents rely only on local information from their neighbors [5].

One of the subfields of MAS are networked cyber-physical systems. In these class of systems, there is a strong coupling between cyber processes such as communication and storage, and physical processes such as actuation and sampling. For example, in wireless sensor networks, the sensors are in a remote location and the action is communicated to the actuator over a wireless communication channel. These channels consists of limited bandwidth and as the number of sensors increases effective utilization of these channels is necessary which has given rise to event triggered control [6]. In event triggered control, traditional periodic sampling is replaced with deliberate, opportunistic and aperiodic sampling thereby reducing load on communication and computation resources [5]. One of the most fundamental dynamical systems for which we can envision an event triggered control is an integrator. Many complex dynamic systems can be feedback linearized as double integrators such as mobile robots, power grids, or quad-copters [4], [7]. The work in [8] was one of the very first notes to study distributed event-based consensus where a time dependent event based condition was defined for a central controller and the results were extended to a distributed case.

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Related to this note are [9]–[11]. In particular [9] proposes a time dependent event-triggering function for first and second order dynamics to achieve average velocity consensus. In [10], event-based protocols on single integrator dynamics involving both fixed and switching topology. In addition it provides bounds on the sampling period in terms of eigenvalues of the graph Laplacian. The work [11] investigates the problem of synchronization in complex networks and provides conditions to ensure that infinitely frequent triggering is excluded by providing lower bounds on inter-event times.

Most of the current work on event-based control focuses on time-dependent thresholds instead of state-dependent ones. This approach requires the system to have prior knowledge of the smallest non-zero eigenvalue of the graph laplacian matrix and the event triggered condition (ETC) is updated continuously [12]. This, however, may not be practical when the graph has switching topology or if an agent is disconnected from the system. Also, time dependent thresholds do not take into account the proximity of agents which may result in farther agents communicating more frequently compared to closer ones.

In this work, we study the state-dependent event-triggered control strategy for a continuous time second-order MAS over an undirected network. The main contributions of this paper are as follows. Firstly, a novel state dependent ETC is derived along with sufficient conditions to asymptotically achieve consensus. The derived ETC can be continuous or piecewise continuous, thereby giving us the flexibility to choose a continuous or a piecewise continuous controller. Secondly, we use this ETC to achieve average consensus with a final constant velocity and zero-velocity consensus with a final zero velocity. This is important when you want to convert a consensus problem to a rendezvous problem. As we will see further in this paper, converting a consensus problem to a rendezvous problem can be done very easily by just updating the controller in continuous time while still operating under the same ETC. This is a very strong result and will be the main focus of this paper. Finally, with the simulation results we prove the effectiveness of the derived ETC under both cases and provide a comparison with constant time-scheduled control that was presented in [13].

This paper is organized as follows. Section II presents the background and system model. The proposed event triggered scheme is derived in Section III. Section IV verifies the results with numerical example and simulations. Conclusion is presented in Section V.

Notations: Some concepts of graph theory that will be used through out this paper are defined here [14]. An undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ consists of a finite vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$, representing N agents, and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ where each edge is denoted by $(v_i, v_j) \in \mathcal{E}$. The adjacency matrix of the graph is denoted as $A_{\mathcal{G}} = [a_{ij}]$ is the $N \times N$ matrix given by $a_{ij} = 1$, if $(i,j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The degree matrix $\Delta(\mathcal{G})$ for an undirected, unweighted graph is a diagonal matrix $\mathrm{diag}(d_1, d_2, \dots, d_n)$ with d_i being the number of neighbors of agent i. The Laplacian matrix $L(\mathcal{G})$ associated with the undirected graph $\mathcal{G}$ is defined as $L(\mathcal{G}) = \Delta(\mathcal{G}) - A_{\mathcal{G}}$ where $\Delta(\mathcal{G})$ is the degree matrix and $A_{\mathcal{G}}$ is the adjacency matrix.

II. PROBLEM FORMULATION

We consider multi-agent systems having double integrator dynamics described as,

$$P_i : \{\ \dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t) \qquad i = 1, \dots, N, \quad (1)$$

where $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are position, velocity, and control inputs of agent i, respectively. The agents are able to exchange information over a static and given undirected graph $\mathcal{G}$.

We are interested in distributed control strategies that drive the team of agents to consensus in their state. In this direction, we define two notions of consensus for the multi-agent system comprised of agents with dynamics (1).

Definition 1. A second order system is said to have achieved *average consensus*, if for all $x_i(0),\ \dot{x}_i(0)\ \in \mathbb{R}$, where $i = 1, 2, \dots, N$,

$$\lim_{t\to\infty}\Big(x_i(t) - \frac{t}{N}\sum_{i=1}^{N} v_i(0)\Big) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) \text{ and}$$
$$\lim_{t\to\infty} v_i(t) = \frac{1}{N}\sum_{i=1}^{N} v_i(0).$$

Definition 2. A second order system is said to have achieved *zero velocity consensus*, if for all $x_i(0),\ \dot{x}_i(0)\ \in \mathbb{R}$, where $i = 1, 2, \dots, N$,

$$\lim_{t\to\infty} x_i(t) = c \quad \text{and} \quad \lim_{t\to\infty} v_i(t) = 0,$$

where $c \in \mathbb{R}$ is a scalar constant.

Consider the distributed control law proposed in [15] and [16] and given by

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}[(x_i(t) - x_j(t)) + \mu(v_i(t) - v_j(t))], \quad (2)$$

where a_{ij} is the $\{i,j\}$ entry of the adjacency matrix $A_{\mathcal{G}} \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}$, and μ is a positive scalar. Denote $x(t) = \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^T$ and $v(t) = \begin{bmatrix} v_1 & v_2 & \dots & v_N \end{bmatrix}^T$, then the closed loop dynamics can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \Gamma \begin{bmatrix} x \\ v \end{bmatrix}, \quad \text{where} \quad \Gamma = \begin{bmatrix} 0 & I_N \\ -L & -\mu L \end{bmatrix}.$$

The above system reaches *average consensus* for an appropriate choice of gain μ and a connected graph $\mathcal{G}$, as shown in [15, Lemma 4.1].

The broad gist of an event-triggered scheme is to sample the states $(x(t), v(t))$ as $(\hat{x}(t_k^i), \hat{v}(t_k^i))$ where the sampled states are constant between the event times $\{t_k^i, t_{k+1}^i\}$, that is, a zero-order-hold (ZOH) is applied in between the event times. The error in state measurements for agent i is defined as,

$$\begin{cases} e_x^i(t) &= \hat{x}_i(t_k^i) - x_i(t) \\ e_v^i(t) &= \hat{v}_i(t_k^i) - v_i(t). \end{cases} \quad (3)$$

Agent i broadcasts its latest states to its neighbors at event times $\{t_0^i, t_1^i, \dots, t_k^i\}$ which are decided when an event triggered condition is violated, i.e., when

$$f_i(e_x^i(t), e_v^i(t), \hat{x}_i(t_k^i), \hat{v}_i(t_k^i)) > 0,$$

for some function $f_i(\cdot)$. Such an event triggered condition should be designed to reduce the times at which its states are sampled and communicated to its neighbors, and also decrease its own controller updates thereby reducing the load on its communication and computation resources. Note that in this paper, the states $x(t)$ and $\dot{x}(t)$ are called position and velocity respectively. However, in general, they need not be position and velocity as their definitions can depend on the considered system.

In this direction, we would like to consider an event triggered strategy for implementing the second-order consensus control law in (2).

Problem 1. Construct a state-dependent event triggered condition $f_i(e_x^i(t), e_v^i(t), \hat{x}_i(t_k^i), \hat{v}_i(t_k^i)) > 0$ for the system in (1) and a distributed control for each agent such that the multi-agent systems achieves,

i) *average consensus*,
ii) *zero velocity consensus*.

III. MAIN RESULTS

In this section, we solve the second order multi-agent system consensus problem with the presentation of an event triggered condition (ETC). The general notion of event based sampling is to define a function that guarantees the error function of a system to always be less than some prespecified threshold. Such functions generally take the form below,

$$f(\xi, e) \triangleq g(e) - h(\xi), \qquad \xi = \begin{bmatrix} x \\ v \end{bmatrix}, e = \begin{bmatrix} e_x \\ e_v \end{bmatrix}, \quad (4)$$

where $g(e)$ is some function of the error and $h(\xi)$ is a threshold that in our case is dependent on the states of the system. The proposed scheme closes the loop, as shown in Figure 1, every time the error function crosses the prespecified threshold, i.e., when $f(x, e) = 0$ as defined in (4).

The ETC we propose is defined as follows,

$$f_i(e_x^i(t), e_v^i(t), \hat{x}_i(t), \hat{v}_i(t)) = \Big((e_x^i)^2 + \mu(e_v^i)^2\Big) + \Big(\frac{\sigma\alpha_i}{d_i}(-\mu + d_i\alpha_i + \mu d_i\alpha_i)n_i^2\Big), \quad (5)$$

where e_x^i and e_v^i are sampling errors defined in (3), d_i is the number of neighbors of agent i, α_i and σ are positive constants, n_i is the $\{i\}$ element of $n(t) = Lv(t)$. We now use the ETC in (5) to achieve *average consensus* and *zero-velocity consensus* as defined in Section II.

A. Event Triggered Control for Average Consensus

To minimize the controller updates, the controller in (2) is modified as,

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}[(\hat{x}_i(t_k^i) - \hat{x}_j(t_{k'}^j)) + \mu(\hat{v}_i(t_k^i) - \hat{v}_j(t_{k'}^j))], \quad (6)$$

for $t \in [t_k^i, t_{k+1}^i)$ and $k'(t) = \arg\min_{l\in\mathbb{N}:t\geq t_l^j}\{t - t_l^j\}$ denotes the last event instant of agent j. The sequence $\{t_k\}_{k\in\mathbb{N}}$ are the time instants when agent i samples its states, while agent j samples its states at $\{t_{k'}^j\}_{k'\in\mathbb{N}}$, which says that the event times can be asynchronous, i.e., agent i can sample its states without receiving/requesting its neighbors to sample their states and continues to evolve with the last received information from its neighbors. However, agent i will update its controller as soon as it receives any new information from its neighbors. The exchange of information for the system (1) with controller (6) is shown in Figure 1.

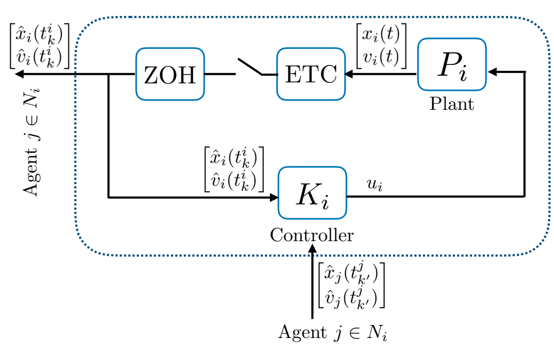


Fig. 1. Agent i with event triggered control law (6).

In Figure 1, the event-triggered condition (ETC) has access to the true state $\{x_i, v_i\}$ of the plant and when the ETC condition is violated, the state is sampled and provided to the controller ($\{\hat{x}_i, \hat{v}_i\}$). These states are available in a sample and hold fashion until the next updated states arrive. We now provide a result showing that the trigger condition (5) with control law (6) achieves average consensus for the system.

Theorem 1. *Consider the system in* (1) *with control* (6), *and assume that the communication graph is connected and undirected. Suppose the event-triggered condition is given by*

$$(e_x^i)^2 + \mu(e_v^i)^2 \geq -\frac{\sigma\alpha_i}{d_i}(-\mu + d_i\alpha_i + \mu d_i\alpha_i)n_i^2, \quad (7)$$

where e_x^i and e_v^i are sampling errors defined in (3), *$n_i(t) = \sum_{j\in N_i}(v_i(t) - v_j(t))$, and $\mu > 0\ \forall i$. Then for $0 < \sigma < 1$, $0 < \alpha_i < \frac{\mu}{d_i + \mu d_i}$ and for any initial conditions, the system* (1) *with control* (6) *achieves consensus.*

Proof. Consider the candidate Lyapunov function $V(x, v) = \frac{1}{2}x^T L^2 x + \frac{1}{2}v^T L v$ where L is symmetric positive semi-definite Laplacian matrix corresponding to the graph $\mathcal{G}$. Then the time derivative of $V(x, v)$ can be expressed as,

$$\begin{aligned}
\dot{V}(x, v) &= x^T L^2 \dot{x} + v^T L \dot{v} = -\mu n^T n - n^T L(e_x + \mu e_v) \\
&= -\mu n^T n - \sum_i \sum_{j\in d_i} n_i(e_x^i - e_x^j) - \mu \sum_i \sum_{j\in d_i} n_i(e_v^i - e_v^j) \\
&= -\mu n^T n - \sum_i d_i n_i e_x^i + \sum_i \sum_{j\in d_i} n_i e_x^j - \mu \sum_i d_i n_i e_v^i + \\
&\quad \mu \sum_i \sum_{j\in d_i} n_i e_v^j.
\end{aligned}$$

Using Young's inequality [17] which states that for a given $x, y \in \mathbb{R}$ and for any $\epsilon \in \mathbb{R}_{>0}$, $|xy| \leq \frac{x^2}{2\epsilon} + \frac{\epsilon y^2}{2}$, we can bound $\dot{V}$ as,

$$\begin{aligned}
\dot{V} &\leq -\mu \sum_i n_i^2 + \Big(\sum_i \frac{d_i\alpha_i n_i^2}{2} + \sum_i \frac{d_i(e_x^i)^2}{2\alpha_i}\Big) \\
&\quad + \Big(\sum_i \frac{d_i\alpha_i n_i^2}{2} + \sum_i \sum_{j\in d_i} \frac{(e_x^j)^2}{2\alpha_i}\Big) \\
&\quad + \mu\Big(\sum_i \frac{d_i\alpha_i n_i^2}{2} + \sum_i \frac{d_i(e_v^i)^2}{2\alpha_i}\Big) \\
&\quad + \mu\Big(\sum_i \frac{d_i\alpha_i n_i^2}{2} + \sum_i \sum_{j\in d_i} \frac{(e_v^j)^2}{2\alpha_i}\Big) \\
&= \sum_i (-\mu + d_i\alpha_i + \mu d_i\alpha_i)n_i^2 + \sum_i \frac{d_i}{\alpha_i}\Big((e_x^i)^2 + \mu(e_v^i)^2\Big).
\end{aligned}$$

Enforcing the below condition,

$$(e_x^i)^2 + \mu(e_v^i)^2 \leq -\frac{\sigma\alpha_i}{d_i}(-\mu + d_i\alpha_i + \mu d_i\alpha_i)n_i^2 \quad (8)$$

ensures that,

$$\dot{V} \leq \sum_i \Big((1 - \sigma)(-\mu + d_i\alpha_i + \mu d_i\alpha_i)n_i^2\Big), \quad (9)$$

which is negative semi-definite $\forall\ \{x, v\}$ given $0 < \sigma < 1$ and $0 < \alpha_i < \frac{\mu}{d_i + \mu d_i}$. With the above Lyapunov function, we now can define the compact set $\emptyset\mathbb{1}mega = \{(x, v)\,|\,V(x, v) \leq c\}$ and let $S = \{(x, v) \in \Omega | \dot{V}(x, v) = 0\}$. Note that $\dot{V} \equiv 0$ in (9) is possible only when $n_i = 0$ meaning that the velocities are in agreement (i.e., $v \in \text{span}\{\mathbb{1}\}$). We now show that S cannot contain any trajectories where $Lx \neq 0$. To prove this by contradiction, let there be an agent $k \in N_i$ such that $x_k > x_i$ which ensures $Lx \neq 0$. Then

$$\begin{aligned}
\dot{v}_i &= -\sum_{j=1}^{n} a_{ij}[(\hat{x}_i - \hat{x}_j) + \mu(\hat{v}_i - \hat{v}_j)] = -\sum_{j=1}^{n} a_{ij}(\hat{x}_i - \hat{x}_j) \\
&> -a_{ik}(\hat{x}_i - \hat{x}_k) > 0,
\end{aligned}$$

due to our assumption that $x_k > x_i$, which is a contradiction because any trajectory in S must have $\dot{v}_i = 0$. Therefore, S must be of the form $S = \{(x, v) \in \Omega \cap (\mathrm{span}\{\mathbb{1}\}, \mathrm{span}\{\mathbb{1}\})\}$. Invoking LaSalle's invariance principle [18], we conclude that all trajectories starting in Ω must converge to S as $t \to \infty$. □

When the error grows and equality is attained in (8), an event is triggered. At this instant, $x_i(t) = x_i(t_k^i)$ and $v_i(t) = v_i(t_k^i)$ thus resetting the error function to 0. This process of error function evolving from 0 to a positive value continues until consensus is achieved. Notice that even though the vector $n(t)$ is a function of continuous time 't', the neighbors provide agent i only with the sampled states $\{x_j(t_l^j), v_j(t_l^j)\}_{l\in\mathbb{N}}$ and we can either set the ETC to receive $x_i(t)$ or $x_i(t_k^i)$ without undermining the results. This is explored in the sequel.

We now take a moment to talk about the error dynamics as defined in (7). In vector form we can re-write (7) as,

$$e_x + \mu e_v \geq \beta(Lv)^2$$

where $e_x = \begin{bmatrix}(e_x^1)^2 & \dots & (e_x^N)^2\end{bmatrix}^T$, $e_v = \begin{bmatrix}(e_v^1)^2 & \dots & (e_v^N)^2\end{bmatrix}^T$, and $\beta = \mathrm{diag}(\beta_1, \dots, \beta_N)$ with $\beta_i = -\frac{\sigma\alpha_i}{d_i}(-\mu + d_i\alpha_i + \mu d_i\alpha_i)$. Then when consensus is achieved, $v = c\mathbb{1}$ with c being a constant. We obtain $e_x + \mu e_v \leq \beta(Lv)^2 = c^2\beta(L\mathbb{1})^2 = 0$ because $L\mathbb{1} = 0$. As the error function is always non-negative, we have $e_x + \mu e_v = 0$. Concluding that when consensus is achieved the error function goes to 0.

B. Event Triggered Control for Zero Velocity Consensus

To achieve zero velocity consensus, we propose a modification to the controller proposed in (6). In this case, we provide both the ETC and the controller access to continuous time self-states $\{x_i(t), v_i(t)\}$, which renders the controller to be continuous as opposed to that presented in III-A which is piecewise continuous. The control is now defined as,

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}[(x_i(t) - \hat{x}_j(t_{k'}^j)) + \mu(v_i(t) - \hat{v}_j(t_{k'}^j))], \tag{10}$$

where $\{\hat{x}_j(t_{k'}^j), \hat{v}_j(t_{k'}^j)\}$ are the last received states from the neighboring agent j. The control in (10) can be represented in vector form as,

$$u(t) = -(\Delta(\mathcal{G})x(t) - A_{\mathcal{G}}\hat{x}(t_{k'}^j)) - \mu(\Delta(\mathcal{G})v(t) - A_{\mathcal{G}}\hat{v}(t_{k'}^j)).$$

Theorem 2. *Consider the system in* (1) *with control* (10)*, and assume that the communication graph is connected and undirected. Suppose the event-triggered condition is given by* (7)*, then for* $0 < \sigma < 1$, $0 < \alpha_i < \frac{\mu(2-\sigma)}{d_i(1+\mu)(1-\sigma)}$ *and for any initial conditions, the system achieves consensus.*

Proof. Consider the candidate Lyapunov function $V(x, v) = \frac{1}{2}x^TL^2x + \frac{1}{2}v^TLv$ where L is a symmetric positive semi-definite Laplacian matrix corresponding to the graph $\mathcal{G}$. Then we have,

$$\begin{aligned}\dot{V}(x, v) &= x^TL^2\dot{x} + v^TL\dot{v}\\ &= x^TL^2v + v^TL[-(\Delta(\mathcal{G})x - A_{\mathcal{G}}\hat{x}) - \mu(\Delta(\mathcal{G})v - A_{\mathcal{G}}\hat{v})]\\ &= -\mu v^TL^2v + v^TLA_{\mathcal{G}}e_x + \mu v^TLA_{\mathcal{G}}e_v\\ &= -\mu\sum_i n_i^2 + \sum_i n_i \sum_{j\in d_i} e_x^j + \mu\sum_i n_i \sum_{j\in d_i} e_v^j.\end{aligned}$$

Using Young's inequality to bound the above equation and enforcing the condition (7), we have,

$$\dot{V}(x, v) \leq \sum_i \Big(\mu(\sigma - 2) + \alpha_i d_i(1 - \sigma) + \mu d_i\alpha_i(1 - \sigma)\Big)\frac{n_i^2}{2},$$

which is negative semi-definite for $0 < \alpha_i < \frac{\mu(2-\sigma)}{d_i(1+\mu)(1-\sigma)}$ and $0 < \sigma < 1$. We have $\dot{V} \equiv 0$ if and only if $n_i = 0$ which is possible only when the velocities are in agreement (i.e., $v \in \mathrm{span}\{\mathbb{1}\}$), hence using LaSalle's invariance principle [18] and the similar arguments posed at the end of Theorem 1, we can conclude that consensus is achieved i.e. $|x_i - x_j| \to 0$ and $|v_i - v_j| \to 0$ as $t \to \infty$. □

Using the preceding theorem, we'd like to determine that the consensus achieved is indeed zero-velocity consensus. In this direction, consider the control in (10) and denote,

$$\xi_i = [x_i \ v_i]^T \quad \text{and} \quad \hat{\xi}_i = [\hat{x}_i \ \hat{v}_i]^T,$$

leading to

$$\dot{\xi}_i = C\xi_i - BK\sum_{j\in d_i} a_{ij}(\xi_i - \hat{\xi}_j) - BF\sum_{j\in d_i} a_{ij}(\xi_i - \hat{\xi}_j), \tag{11}$$

where $C = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $B = \begin{bmatrix}0\\ 1\end{bmatrix}$, $K = \begin{bmatrix}0 & \mu\end{bmatrix}$ and $F = \begin{bmatrix}1 & 0\end{bmatrix}$. Then in state space form, the system can be represented as,

$$\dot{\xi}(t) = \Xi_{\mathcal{G}}\ \xi(t) + \Theta_{\mathcal{G}}\ \hat{\xi}(t) \tag{12}$$

where

$$\Xi_{\mathcal{G}} = I_N \otimes C - \Delta(\mathcal{G}) \otimes B(K + F),\ \Theta_{\mathcal{G}} = A_{\mathcal{G}} \otimes B(K + F).$$

Lemma 3. *The matrix* $\Xi_{\mathcal{G}}$ *is Hurwitz and* $exp(\Xi_{\mathcal{G}}t) \to 0$ *as* $t \to \infty$.

Proof. Denoting the eigenvalues of $\Xi_{\mathcal{G}}$ as $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{2N}$ and expressing the matrix $\Xi_{\mathcal{G}}$ as,

$$\begin{aligned}\Xi_{\mathcal{G}} &= I_N \otimes C - \mathrm{diag}\{d_1, d_2, \dots, d_N\} \otimes (BK + BF)\\ &= \mathrm{diag}\{C - d_1BK - d_1BF, C - d_2BK - d_2BF,\\ &\quad \dots, C - d_NBK - d_NBF\}.\end{aligned}$$

The above matrix $\Xi_{\mathcal{G}}$ is a block diagonal matrix and we can study its individual blocks to gain more perspective on its spectral properties. In particular,

$$\det(C - d_iBK - d_iBF) = \det\left(\begin{bmatrix}0 & 1\\ -d_i & -\mu d_i\end{bmatrix}\right) = d_i \neq 0,$$

which shows that $\text{rank}(C - d_iBK - d_iBF) = 2$, it follows that,

$$\text{rank}(\Xi_{\mathcal{G}}) = \sum_{i=1}^{N} \text{rank}(C - d_iBK - d_iBF) = 2N,$$

hence the matrix $\Xi_{\mathcal{G}}$ is full rank and consists of only non-zero eigenvalues. The characteristic equation corresponding to the matrix $(C - d_iBK - d_iBF)$ for $i = 1, \ldots, N$ is,

$$f(s) = \begin{vmatrix} -s & 1 \\ -d_i & -\mu d_i - s \end{vmatrix} = s^2 + (\mu d_i)s + d_i,$$

and the roots of the above polynomial are,

$$s = \frac{-\mu d_i \pm \sqrt{\mu^2 d_i^2 - 4d_i}}{2},$$

$\forall \mu, d_i > 0$, the matrices $C - d_iBK - C_iBF$ for $i = 1, 2, \ldots, N$ are Hurwitz stable, implying the matrix $\Xi_{\mathcal{G}}$ is Hurwitz. Since $\Xi_{\mathcal{G}}$ is Hurwitz, $\exp(\Xi_{\mathcal{G}}t) \to 0$ as $t \to \infty$. ☐

Theorem 4. *Consider a connected and undirected graph $\mathcal{G}$, then under the event triggered condition* (7) *control in* (10) *achieves zero velocity consensus.*

Proof. From Theorem 2 we know that the system achieves consensus i.e., $x_i(t) \to x_j(t)$ and $v_i(t) \to v_j(t)$. Now to prove that this is a zero-velocity consensus, consider the general solution to the state space equation in (12) as,

$$\begin{aligned}
\xi(t) &= \exp(\Xi_{\mathcal{G}}t)\xi(0) + \exp(\Xi_{\mathcal{G}}t) \int_0^t \exp(-\Xi_{\mathcal{G}}\tau)\Theta_{\mathcal{G}}\hat{\xi}(\tau)d\tau \\
&= \exp(\Xi_{\mathcal{G}}t)\xi(0) + \exp(\Xi_{\mathcal{G}}t)\Big(- \exp(-\Xi_{\mathcal{G}}\tau)\Xi_{\mathcal{G}}^{-1}\Theta_{\mathcal{G}}\hat{\xi}(\tau)\Big|_0^t \\
&\quad - \int_0^t \exp(-\Xi_{\mathcal{G}}\tau)(-\Xi_{\mathcal{G}})^{-1}\Theta_{\mathcal{G}}\dot{\hat{\xi}}(\tau)d\tau\Big) \\
&= \exp(\Xi_{\mathcal{G}}t)\xi(0) - \Xi_{\mathcal{G}}^{-1}\Theta_{\mathcal{G}}\hat{\xi}(t) + \exp(\Xi_{\mathcal{G}}t)\Xi_{\mathcal{G}}^{-1}\Theta_{\mathcal{G}}\hat{\xi}(0) \\
&\quad + \exp(\Xi_{\mathcal{G}}t) \int_0^t \exp(-\Xi_{\mathcal{G}}\tau)\Xi_{\mathcal{G}}^{-1}\Theta_{\mathcal{G}}\dot{\hat{\xi}}(\tau)d\tau.
\end{aligned}$$

For a very large t, since $\Xi_{\mathcal{G}}$ is Hurwitz, $\exp(\Xi_{\mathcal{G}}t) \to 0$ as $t \to \infty$ and since consensus is achieved, $\dot{\hat{v}} \to 0$ and the above equation reduces to,

$$\lim_{t\to\infty} \xi(t) = \lim_{t\to\infty} -\Xi_{\mathcal{G}}^{-1}\Theta_{\mathcal{G}}\hat{\xi}(t). \tag{13}$$

The matrix $\Xi_{\mathcal{G}}$ is a block diagonal matrix with blocks $\begin{bmatrix} 0 & 1 \\ -d_i & -\mu d_i \end{bmatrix}$ as shown in Lemma 3 and it's easy to see that it's inverse is also a block diagonal matrix with blocks $\begin{bmatrix} -\mu & -1/d_i \\ 1 & 0 \end{bmatrix}$. The matrix $\Theta_{\mathcal{G}}$ is also composed of blocks with components $\begin{bmatrix} 0 & 0 \\ \times & \times \end{bmatrix}$ and the product of $\Xi_{\mathcal{G}}^{-1}$ and $\Theta_{\mathcal{G}}$ will be matrices with blocks of $\begin{bmatrix} \times & \times \\ 0 & 0 \end{bmatrix}$. Hence, we see that all the even numbered rows of (13) which correspond to the velocities of the agents are 0 as $t \to \infty$. Therefore $\lim_{t\to\infty} v_i(t) = 0$ and it's obvious that $\lim_{t\to\infty} x_i(t) =$ constant. ☐

Reducing the communication load in a network comes at a cost of slower convergence rate. To prove this, define the convergence rate of $V(x, v)$ as $\rho := \frac{1}{2} \inf \Big\{ -\frac{\dot{V}(x,v)}{V(x,v)} : (x, v) \in S \Big\}$ [19]. In this expression, infimum is attained when $V(x, v)$ and $V(x_0, v_0)$ are at its maximum which is $V(x_0, v_0)$ and $\dot{V}(x_0, v_0)$ respectively. Let ρ_e be the convergence rate with event-based sampling and ρ_w without event-based sampling and our goal is to show $\rho_e < \rho_w$. Then,

$$\rho_e = \frac{\sum_i (1 - \sigma)(\mu - d_i\alpha_i - \mu d_i\alpha_i)n_i(0)^2}{\frac{1}{2}x_0^T L^2 x_0 + \frac{1}{2}v_0^T L v_0}, \tag{14}$$

and,

$$\rho_w = \frac{\mu v_0^T L^2 v_0}{\frac{1}{2}x_0^T L^2 x_0 + \frac{1}{2}v_0^T L v_0}. \tag{15}$$

Maximizing ρ_e over α_i we set $\alpha_i = 0$ to obtain $\rho_e = \frac{(1-\sigma)\mu \sum_i n_i(0)^2}{\frac{1}{2}x_0^T L^2 x_0 + \frac{1}{2}v_0^T L v_0}$ and it can be shown that $\sum_i n_i(0)^2 = v_0^T L^2 v_0$. Therefore, $\rho_e = (1 - \sigma)\rho_w$ implying $\rho_e < \rho_w$. This also holds for the zero-velocity consensus case.

IV. Simulation Example

In this section, we illustrate the theoretical results through simulations for a network of 4 agents with communication graph $\mathcal{G}$ as shown in Figure (2).

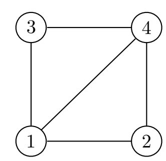


Fig. 2. Communication graph $\mathcal{G}$ of the multi-agent system.

Average consensus: Simulation results for double-integrator agents with random initial conditions, $\mu = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.1$ and $\sigma = 0.9$ are shown in Figure 3. The results in Figure 3 are consistent with Theorem 1 as all the agents reach position consensus with constant final velocity. Since the control law is piecewise continuous, this event based scheme makes use of lesser computing power compared to traditional periodic sampling. The evolution of error and trigger functions for agent 1 is shown in Figure 4(a). As discussed earlier, the error function increases from 0 in the positive direction until equality is attained in (8) and subsequently the error function is reset to zero and this process repeats until consensus is achieved. The threshold function, as indicated in Figure 4(a) is re-evaluated any time new information is received from the neighbors. Figure 4(b) shows the evolution of the Lyapunov function as defined in Theorem 1 and we note that $V(x, v) \to 0$ as consensus is achieved.

Zero velocity consensus: The simulation results for second order multi agent system with random initial conditions, $\mu = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.1$ and $\sigma = 0.9$ is shown in Figure 5 where we conclude that the velocity convergence takes place at zero and position consensus takes place at a constant value consistent with the results of Theorem 4. The

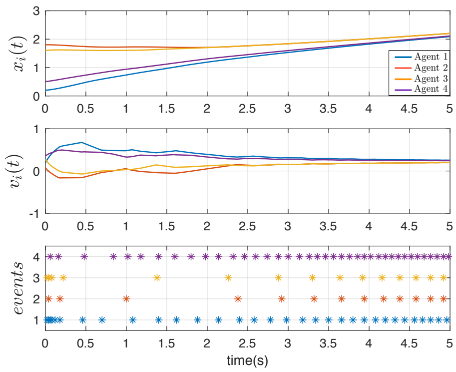
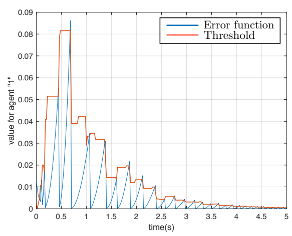
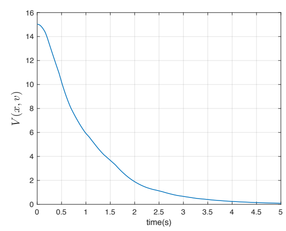


Fig. 3. Simulation result for double-integrator agents with trigger function (5) and control law (2).



(a) Error function and the trigger function for agent 1



(b) Evolution of the Lyapunov function.

Fig. 4. Results pertaining to the average consensus problem.

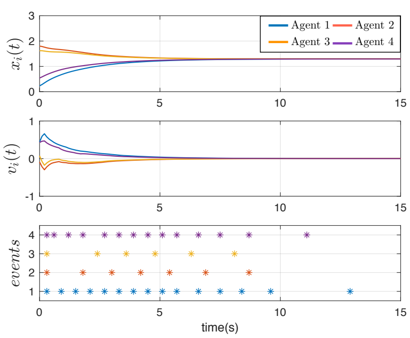


Fig. 5. Simulation result for double-integrator agents with trigger function (5) and control law (6).

controller inputs to individual agents as defined in equation (10) is shown in Figure 6 and $u_i \to 0$ as the states converge.

V. Concluding Remarks

In this paper, a novel event triggered condition (ETC) was proposed to asymptotically achieve average and zero-velocity consensus among a group of agents with second-order dynamics. We highlight that the proposed condition relies only on relative state measurements and reduces the communication load on the system. The proposed event triggered scheme determines for each agent when to update its own states and when to broadcast its states when a locally computed error function exceeds a state-dependent threshold. The simulations were performed with periodic event detection which further minimizes the communication load on the system and also provides a lower bound on the inter-event time.

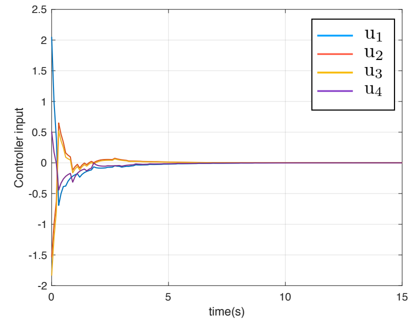


Fig. 6. Controller inputs for zero-velocity consensus.

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